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13. ABSTRACT

The work of Rabin on computable algebra is extended by Cannonito and Gatterdam by applying the Grzegorczyk hierarchy to obtain an improved concept of a computable group. Word problems are shown to be algebraic invariants for computable groups with standard indicies. Higman embedding is covered along with its relationship to the Strong Britton extension. An excellent flow chart is presented to aid the reader in visualizing the relationship the several sections bear to each other.

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49

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THE COMPUTABILITY OF GROUP CONSTRUCTIONS, PART I

F. B. Cannonito[1] and R. W. Gatterdam[2]

The beginnings of computable algebra, especially as this applies to group theory, can be traced for all practical purposes to the fundamental paper of M. O. Rabin [9]. In this work Rabin showed the very natural equivalence between the existence of a "recursive realization" of the group in the natural numbers and the existence of a recursive solution to the word problem with respect to any finitely generated presentation of the group. Of course, the property of having word problem solvable with respect to a given presentation applies, in general, only to finitely generated (henceforth f.g.) presentations; there is no problem in presenting even f.g. groups on infinitely many generators so that the word problem with respect to such a presentation is unsolvable.

Subsequently Cannonito sharpened the concept of a computable group in [1] by superimposing the Grzegorczyk hierarchy, denoted "$(\mathcal{E}^{\alpha})_{\alpha\in\omega}$", and by showing that f.g. $\mathcal{E}^{\alpha}$-computable groups existed in profusion for infinitely many consecutive levels of the hierarchy. Although incorrectly stated in [1], it was shown in Cannonito [1], [2] that with respect to standard indices of $\mathcal{E}^{\alpha}$ groups, the word problem is an algebraic invariant; that is, every f.g. presentation of an $\mathcal{E}^{\alpha}$-computable group has, with respect to a standard index, a word problem solvable by an $\mathcal{E}^{\alpha}$ function if one f.g. presentation of the group possesses this property.

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However, certain inelegant aspects of the theory of $\mathcal{E}^{\alpha}$ computable groups came to light as a consequence of the discovery that the proof in [1] of the equivalence of $\mathcal{E}^{\alpha}$ computable realizations and $\mathcal{E}^{\alpha}$ solvable word problems only applies to standard indices. These aspects are discussed, among other parts of the theory, in Cannonito [2]. Typical among the inelegancies is the inability, in general, to say more than this: if a f.g. group has an $\mathcal{E}^{\alpha}$ realization then its word problem is not higher than $\mathcal{E}^{\alpha+1}$. Furthermore, while the assertion in [1] claiming the groups G_{α} presented as

$$\langle a,b,c,d; (a^n b^n)^n = (c^n d^n)^n, n\in U_{\alpha}\rangle$$

where U_{α} is a set of natural numbers decidable only at $\mathcal{E}^{\alpha}$ or higher, had word problem solvable at level $\mathcal{E}^{\alpha}$ but not lower was true, the proof given was flawed in an essential detail; namely, contrary to the assertion in [1], Theorem 6.1, the best that can be said for the free product with amalgamation of $\mathcal{E}^{\alpha}$ groups (under suitable $\mathcal{E}^{\alpha}$ conditions on subgroups and isomorphisms, etc.) is that its level of computability is no higher than $\mathcal{E}^{\alpha+1}$. Thus applying the analysis in [1], to the groups G_{α}, we see the word problem is solvable no higher than $\mathcal{E}^{\alpha+2}$ and no lower than $\mathcal{E}^{\alpha}$. However, in this paper we show the word problem of the G_{α} actually resides no higher than $\mathcal{E}^{\alpha}$, owing to the fact that the G_{α} arise from a rather restricted type of free product with amalgamation in which both factors are copies and both subgroups to be amalgamated are also copies and the isomorphisms are the identities. This situation and its consequences are examined in detail in this paper with respect to a "relativized Grzegorczyk hierarchy" which seems to be appropriate for all countable groups whether finitely generated or not and whether possessed of a solvable word

problem for f.g. presentations or not. Owing to the possible independent interest in this relative Grzegorczyk hierarchy we begin this paper with a discussion and proofs of this concept, which resembles the notion of relative computability as exposed in Davis [3], the essential idea being to close a level $\mathcal{E}^{\alpha}$ under the usual operations after first adding the characteristic function of some subset of the natural numbers.

In the next phase of the theory of $\mathcal{E}^{\alpha}$ groups the Higman embedding [6] of recursively presented groups into finitely presented (f.p.) groups was studied in his doctoral dissertation by Gatterdam [4]. It was shown there that the Higman embedding took f.g. groups with primitive recursive (p.r.) realization into f.p. groups which have a p.r. realization. Moreover, it was shown by Gatterdam that the actual embedding function was elementary (or, equivalently, $\mathcal{E}^{3}$) in the solution to the word problem of the receiver group. Hence, since the groups G_{α} are all recursively presented, they can be effectively embedded in f.p. p.r. groups and so there is an infinite spread of complexity of f.p. groups with respect to the $\mathcal{E}^{\alpha}$ hierarchy. But ideally one wants to show the Higman embedding actually preserves the relative Grzegorczyk hierarchy because by so doing one by product would be the possibility to locate precisely on the hierarchy the receiver groups of the G_{α}. According to Gatterdam's analysis all that could be said is that the receiver groups must slip arbitrarily high up the hierarchy, but conceivably there could be great gaps and no method was at hand to locate the actual level of the hierarchy at which any particular receiver group resided. The reason for this dilemma can be found in a particular construction needed for the Higman proof which has come to be known as the Strong Britton extension. The situation is this: we start with a f.g. group G and an isomorphism

$H < G$ into G itself. If G has presentation, say,

$$\langle a_1,\cdots,a_n;\ R_1,\cdots\rangle$$

then we obtain a new group G_{φ} which embedds G with presentation

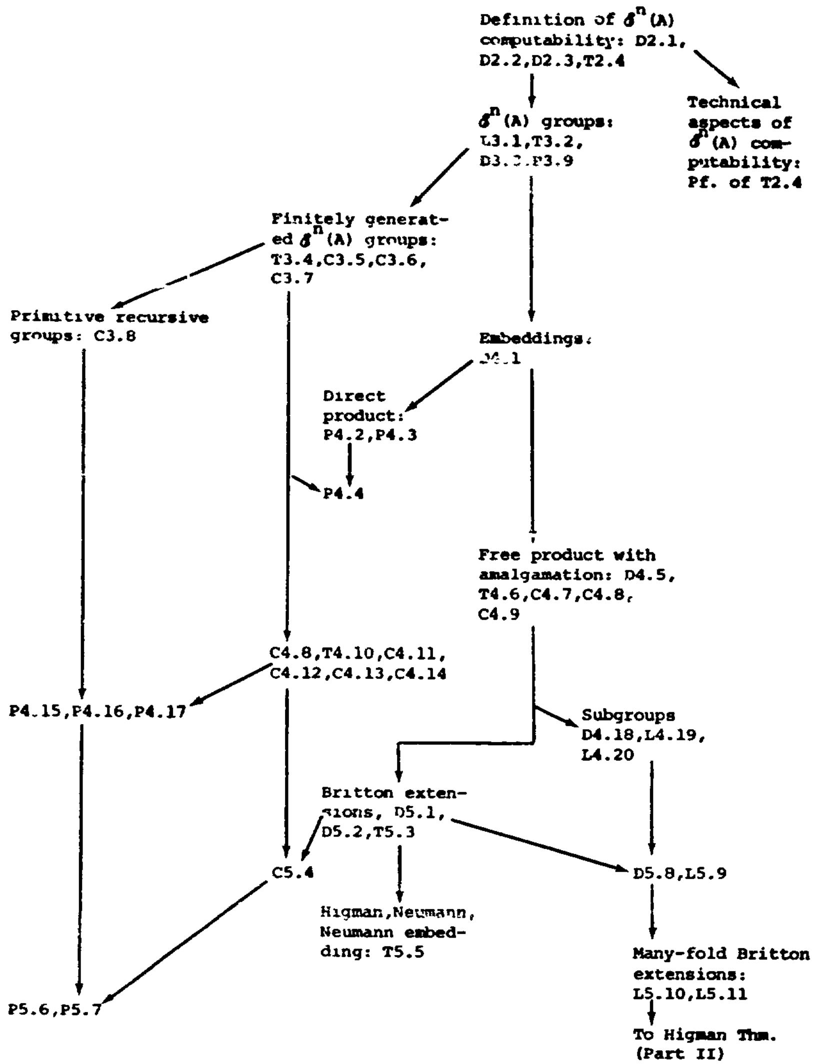
$$\langle a_1,\cdots,a_n,t\ ;\ R_1,\cdots,tht^{-1} = \varphi(h) \quad \text{for all} \quad h\in H\rangle.$$

Thus, G_{φ} is obtained by adjoining a (distinct) infinite cycle t which gives the isomorphism φ by conjugation. Since G_{φ} is a f.g. subgroup of a particular free product with amalgamation the computability of G_{φ} can be shown to lie no higher $\mathcal{E}^{\alpha+1}$ when G lies at level $\mathcal{E}^{\alpha}$. But the Higman embedding uses a countable number of such extensions resulting in a manyfold extension $G_{\varphi_1\varphi_2\varphi_3}\cdots$ or, more simply, G_{∞}. Thus the level of computability can slip extremely high since it seems to require jumping possibly two levels for each new infinite cycle adjoined. However, in this work we show (with respect to the relativized hierarchy) that when G is $\mathcal{E}^{\alpha}$ computable then G_{∞} is at most at level $\mathcal{E}^{\alpha+2}$. Thus, at this state of science the Higman embedding seems to actually preserve the relativized heirarchy from some point on for the remaining constructions do not appear to be troublesome at all. This will be the subject of Part II of this paper, to appear later, and will be based on the analysis we give in the present paper.

Additional constructions studied in this paper form the main stock in trade of infinite group theory. Among these are the "HNN extension" due to Higman, Neumann and Neumann. Also we show there can be no universal primitive recursive group containing a copy, even, of all f.p. p.r. groups

For the convenience of the reader we give a flow chart showing the relationship the several sections bear to each other. The expressions "D2.3,

L3.5, P2.6, and T2.7" refer respectively to Definition 2.3, Lemma 3.5, Proposition 2.6 and Theorem 2.7.



We will observe the following conventions: "ω" denotes the natural numbers $0,1,2,\cdots$ and unmodified "integer" usually means natural number, exceptions to be explicitly mentioned in the context. We omit parentheses as much as possible, particularly in forming the composition of functions; thus, "$fg(x)$" rather than "$f(g(x))$". The characteristic function c_A of a subset A of ω is 0 on A and 1 off A. The notation "$x \longmapsto f(x)$" or "$x \overset{f}{\longmapsto} y$" give the actual value assigned to x under the mapping f. We usually omit the f when this will cause no confusion. Finally, the n-tuple $(x_1,\cdots x_n)$ is abbreviated to $x^{(n)}$ when it is not necessary to explicitly give the coordinates. All other notation will be defined <u>in situ</u>.

§2. Relative Grzegorczyk hierarchy

The purpose of this section is to define a relative Grzegorczyk hierarchy and to verify certain properties of this hierarchy are retained in the relative version. Our point of departure is the hierarchy defined by Grzegorczyk [5], but we modify the definition and relativize with respect to an arbitrary subset of the natural numbers $A \subset N$. The modifications are to use the characterization of the Grzegorczyk hierarchy due to Ritchie, [10] and replace recursion by iteration plus additional initial functions in the manner of Robinson, [11]. We relativize by adjoining the characteristic function c_A to the initial functions similar to Davis' definition of relative recursion and relative primitive recursion, [3]. Essential use is made of Ritchie, [10].

Definition 2.1: A function $f: N^{k+1} \to N$ is defined by limited recursion from functions $g: N^k \to N$, $h: N^{k+2} \to N$ and $j: N^{k+1} \to N$ if it is given by the schema

$$f\left(x^{(k)}, 0\right) = g\left(x^{(k)}\right)$$

$$f\left(x^{(k)}, y+1\right) = h\left(x^{(k)}, y, f\left(x^{(k)}, y\right)\right)$$

subject to the condition

$$f\left(x^{(k)}, y\right) \leq j\left(x^{(k)}, y\right).$$

The function $f: N \to N$ is defined from $h: N \to N$ and $j: N \to N$ by

limited iteration if it is defined by the schema:

$$f(0) = c$$

$$f(x+1) = hf(x)$$

subject to the condition

$$f(x) \leq j(x).$$

Of course the word "limited" in the definition above refers to the bounding functions j. Omitting the bounds j one has the definition of primitive recursion and iteration.

Following Richie, we use the "pairing" functions

$$J(x,y) = \left((x+y)^2 + x\right)^2 + y$$

$$K(z) = E\left([z^{\frac{1}{2}}]\right) = [z^{\frac{1}{2}}] \dot{-} \left[[z^{\frac{1}{2}}]^{\frac{1}{2}}\right]$$

$$L(z) = E(z) = z \dot{-} [z^{\frac{1}{2}}]$$

where $[z^{\frac{1}{2}}]$ = largest integer whose square is less than z. The important properties of these functions are $KJ(x,y) = x$ and $LJ(x,y) = y$.

Inductively for $n \geq 2$ and $J = J^{(2)}$, $M_1^{(2)} = K$, $M_2^{(2)} = L$ we define

$$J^{(n+1)}\left(x^{(n+1)}\right) = J\left(J^{(n)}\left(x^{(n)}\right), x_{n+1}\right)$$

$$M_i^{(n+1)}(z) = M_i^{(n)}K(z) \text{ for } 1 \leq i \leq n$$

$$M_{n+1}^{(n+1)}(z) = L(z).$$

Then the relation between recursion and iteration is given by Theorem 3.2 of [10].

Theorem. Let $f(x^{(n)}, y)$ be defined by primitive recursion

$$f(x^{(n)}, 0) = g(x^{(n)}),\ f(x^{(n)}, y+1) = h(x^{(n)}, y, f(x^{(n)}, y))$$

for $n \geq 0$. Then

$$f(x^{(n)}, y) = \begin{cases} g(0^{(n)}) & \text{if } J^{(n+1)}(x^{(n)}, y) = 0 \\ Lf'J^{(n+1)}(x^{(n)}, y) & \text{otherwise} \end{cases}$$

where f' is defined by the iteration

$$f'(0) = 0$$
$$f'(z+1) = H''f'(z)$$

for

$$H''(w) = J(K(w)+1, H'(K(w), L(w)))$$

$$H'(z,w) = \begin{cases} H(0, g(0^{(n)})) & \text{if } z = 0 \\ H(z,w) & \text{otherwise} \end{cases}$$

$$H(z,w) = \begin{cases} g(M_1^{(n+1)}(z+1), \cdots, M_n^{(n+1)}(z+1)) & \text{if } M_{n+1}^{(n+1)}(z) = 0 \\ h(M_1^{(n+1)}(z), \cdots, M_{n+1}^{(n+1)}(z), w) & \text{otherwise.} \end{cases}$$

The importance of this theorem is that in a class of functions containing the pairing functions J, K, L, $J^{(n)}$, $M_i^{(n)}$ and closed under substitution, closure under iteration implies closure under recursion. Moreover, J being monotone increasing $f(x^{(n)}, y) \leq j(x^{(n)}, y)$ implies $f'(z) \leq J(z, j(M_1^{(n+1)}(z), \cdots, M_{n+1}^{(n+1)}(z)))$ so closure under limited iteration implies closure under limited recursion.

We now define the relative Grzegorczyk hierarchy using the functions $f_n : N^2 \to N$ of Ritchie.

Definition 2.2:

$$f_0(x,y) = x+1$$
$$f_1(x,y) = x+y$$
$$f_2(x,y) = xy$$
$$\left.\begin{array}{l} f_{n+1}(x,0) = 1 \\ f_{n+1}(x,y+1) = f_n(x, f_{n+1}(x,y)) \end{array}\right\} \text{ for all } n \geq 2.$$

Definition 2.3: Let $A \subset N$. Then the class of functions $\mathcal{E}^n(A)$ for $n \geq 2$ (with domain N^k for arbitrary k and Range N) is the smallest class of functions containing the initial functions

$$Z(x) = 0$$
$$U_m^n(x_1, \cdots, x_n) = x_m \qquad \text{for } 1 \leq m \leq n$$
$$f_0(x,y) = x+1$$
$$f_1(x,y) = x+y$$
$$f_n(x,y)$$
$$E(x) = x - [x^{\frac{1}{2}}]$$
$$c_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

and closed under substitution and limited iteration.

Observe that $E(x) \in \mathcal{E}^n(A)$ implies the pairing functions are in $\mathcal{E}^n(A)$ so by the theorem stated above, $\mathcal{E}^n(A)$ is closed under limited recursion. Ritchie's main result is that the usual Grzegorczyk hierarchy is $\mathcal{E}^n = \mathcal{E}^n(N)$. Clearly $\mathcal{E}^n \subset \mathcal{E}^n(A)$ for every A. Also, $f_n(x,y) \in \mathcal{E}^n \subset \mathcal{E}^{n+1} \subset \mathcal{E}^{n+1}(A)$ implies $\mathcal{E}^n(A) \subset \mathcal{E}^{n+1}(A)$.

The following theorem is useful.

<u>Theorem 2.4</u>. If $f: N^k \to N$ is defined by primitive recursion from functions in $\mathcal{E}^n(A)$ for $n \geq 2$ then $f \in \mathcal{E}^{n+1}(A)$.

<u>Proof</u>: By the previous theorem it suffices to consider $f: N \to N$ given by iteration, $f(0) = c = h^0(c)$ and $f(x+1) = hf(x) = h^{x+1}(c)$ (i.e., the composition $h \circ h \circ \cdots \circ h(c)$ $x+1$ times). We must show there exists $j(x) \in \mathcal{E}^{n+1}(A)$ such that $f(x) \leq j(x)$.

The proof given is a modification of the techniques used in the proofs of Theorems 2.2 and 2.3 of [10]. The modifications allow the application of these methods directly to the class $\mathcal{E}^n(A)$ for $n \geq 2$. First we need some facts about $f_n(x,y)$. The proofs which are not included may be found in [10] as indicated or may be supplied by the reader using straightforward induction arguments.

(1) For $n \geq 2$, $f_n(x,1) = x$. (pf inductively $f_{n+1}(x,1) = f_n\left(x, f_{n+1}(x,0)\right) = f_n(x,1)$ & $f_2(x,1) = x$).

(2) For $n \geq 1$, $x \geq 2$, $f_n(x,y)$ is a strictly monotonic increasing function of y. (Lemma 1.1 of [10]).

(3) For $y \geq 1$ and all n, $f_n(x,y)$ is a strictly monotonic increasing function of x. (Lemma 1.2 of [10]).

(4) For $x \geq 2, y \geq 0$ and $n \geq 2$, $f_n(x,y) \leq f_{n+1}(x,y)$. (Lemma 1.3 of [10]).

(5) For $p \geq 2, n \geq 3, x \geq 2$, and all q, $f_n\left(f_n(x,p),q\right) \leq f_n\left(x, f_{n-1}(p,q)\right)$. (Theorem 1.1 of [10]).

(6) For $m > n \geq 2, x \geq 2$ and all y,z, $f_n\left(f_m(x,y), f_m(x,z)\right) \leq f_m(x,y+z)$. (Theorem 1.1 of [10]).

(7) For any $f\left(x^{(p)}\right) \in \mathcal{E}^n(A)$ with $n \geq 2$, there exists $k > 0$, such that when $x_i \geq 2$ for all $1 \leq i \leq p$, $f\left(x^{(p)}\right) < f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), k\right)$.

<u>Proof of (7)</u>: We show that the desired property holds for the initial functions and is preserved under substitution and limited iteration. For $x, y, x_i \geq 2$ we have:

$$Z(x) = 0 < f_{n+1}(x,1) = x$$

$$U_m^p\left(x^{(p)}\right) = x_m < \mathcal{J}^{(p)}\left(x^{(p)}\right) = f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), 1\right) \qquad \text{for } 0 \leq m \leq p$$

$$f_0(x,y) = x+1 < 2x = f_2(x,2) \leq f_{n+1}(x,2) \leq f_{n+1}\left(\mathcal{J}^{(2)}(x,y), 2\right)$$

$$f_1(x,y) = x+y < 2\mathcal{J}^{(2)}(x,y) = f_2\left(\mathcal{J}^{(2)}(x,y), 2\right) \leq f_{n+1}\left(\mathcal{J}^{(2)}(x,y), 2\right)$$

$$f_n(x,y) \leq f_n\left(\mathcal{J}^{(2)}(x,y), \mathcal{J}^{(2)}(x,y)\right) = f_n\left(\mathcal{J}^{(2)}(x,y), f_{n+1}\left(\mathcal{J}^{(2)}(x,y), 1\right)\right)$$
$$= f_{n+1}\left(\mathcal{J}^{(2)}(x,y), 2\right) < f_{n+1}\left(\mathcal{J}^{(2)}(x,y), 3\right)$$

$$E(x) < x = f_{n+1}(x,1)$$

$$c_A(x) < 2 \leq x = f_{n+1}(x,1) \ .$$

It is easy to verify by induction that $f_3(x,y) = x^y$ and that there exists $r, s > 0$ such that for all $x \geq 0$, $\mathcal{J}^{(q)}(x,\cdots,x) \leq (rx)^s$ (the r and s depend on q). Then $\mathcal{J}^{(q)}(x,\cdots,x) \leq (rx)^s = f_3(rx,s) = f_3\left(f_2(x,r), s\right) \leq f_3\left(f_3(x,r), s\right) \leq f_3\left(x, f_2(r,s)\right)$. Thus for each q there exists k' such that $\mathcal{J}^{(q)}(x,\cdots,x) < f_{n+1}(x,k')$.

Let $f\left(x^{(p)}\right) = h\left(g_1\left(x^{(p)}\right), \cdots, g_q\left(x^{(p)}\right)\right)$ where $h\left(x^{(q)}\right) < f_{n+1}\left(\mathcal{J}^{(q)}\left(x^{(q)}\right), k\right)$ and $g_i\left(x^{(p)}\right) < f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), k_i\right)$ and let $k_0 = \max\{k_1, \cdots, k_q\}$. Then

$$
\begin{aligned}
f\left(x^{(p)}\right) &< f_{n+1}\left(\mathcal{J}^{(q)}\left(g_1\left(x^{(p)}\right), \cdots, g_q\left(x^{(p)}\right)\right), k\right) \\
&\leq f_{n+1}\left(\mathcal{J}^{(q)}\left(f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), k_1\right), \cdots, f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), k_q\right)\right), k\right) \\
&\leq f_{n+1}\left(\mathcal{J}^{(q)}\left(f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), k_0\right), \cdots, f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), k_0\right)\right), k\right) \\
&\leq f_{n+1}\left(f_{n+1}\left(f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), k_0\right), k'\right), k\right) \\
&\leq f_{n+1}\left(f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), f_n\left(k_0, k'\right)\right), k\right) \\
&\leq f_{n+1}\left(\mathcal{J}^{(p)}\left(x^{(p)}\right), f_n\left(f_n\left(k_0, k'\right), k\right)\right).
\end{aligned}
$$

Thus the desired property is preserved under substitution.

If $f(x)$ is defined by limited iteration from $h(x)$ and $j(x)$, then $f(x) \leq j(x) < f_{n+1}(x,k)$ for some k so limited iteration preserves the desired property. Thus we have verified (7).

Returning now to the proof of the theorem let f be defined by $f(0) = c = h^0(c)$ and $f(x+1) = hf(x) = h^{x+1}(c)$ for $h(x) \in \mathcal{E}^n(A)$ and $n \geq 2$.

Then by (7), there exists k such that $h(y) < f_{n+1}(y,k)$ for $y \geq 2$. In particular $h^0(y) = y \leq \max(2,y) = f_{n+1}\left(\max(2,y), 1\right) = f_{n+1}\left(\max(2,y), f_{n+1}(k,0)\right)$ for all y. Inductively assume $h^x(y) \leq f_{n+1}\left(\max(2,y), f_{n+1}(k,x)\right)$ for all y. Then

$$
\begin{aligned}
h^{x+1}(y) = hh^x(y) &\leq f_{n+1}\left(\max\left(2, h^x(y)\right), k\right) \\
&\leq f_{n+1}\left(f_{n+1}\left(\max(2,y), f_{n+1}(k,x)\right), k\right) \\
&\leq f_{n+1}\left(\max(2,y), f_n\left(f_{n+1}(k,x), k\right)\right) \\
&= f_{n+1}\left(\max(2,y), f_n\left(f_{n+1}(k,x), f_{n+1}(k,1)\right)\right) \\
&\leq f_{n+1}\left(\max(2,y), f_{n+1}(k,x+1)\right),
\end{aligned}
$$

where the first line follows from (7), the second line from the inductive hypothesis and $f_{n+1}\left(\max(2,y), f_{n+1}(k,x)\right) \geq 2$, the third line from (5), the fourth line from (1) and the fifth line from (6).

Therefore $f(x) = h^x(c) \leq f_{n+1}\left(\max(2,c), f_{n+1}(k,x)\right) \in \mathcal{E}^{n+1}(A)$ as a function of x. □

Since by the theorem unlimited recursion leads from the class $\mathcal{E}^n(A)$ to the class $\mathcal{E}^{n+1}(A)$, the class $\bigcup_{n=2}^{\infty} \mathcal{E}^n(A)$ is closed under iteration and hence primitive recursion. Thus we have

Corollary 2.5. The class $\bigcup_{n\geq 2}^{\infty} \mathcal{E}^n(A)$ is the class of A-primitive recursive functions, and for all A and each $n \geq 2$, $\mathcal{E}^n(A)$ is properly contained in $\mathcal{E}^{n+1}(A)$. □

The special case $c(A) \in \mathcal{E}^m$ for some m, (i.e., A is $\mathcal{E}^m$ decidable) is of particular interest. Here $\mathcal{E}^n(A) \subset \mathcal{E}^p$ for $p = \max\{n,m\}$. However, suppose $n < m$ so $\mathcal{E}^n(A) \subset \mathcal{E}^m$. Then by the usual estimates an unbounded recursion leads from $\mathcal{E}^n(A)$ to $\mathcal{E}^{m+1}$. However we see from Theorem 4 that such an unbounded recursion leads from $\mathcal{E}^n(A)$ to $\mathcal{E}^{n+1}(A) \subset \mathcal{E}^m$. It should be noted that $c(A) \in \mathcal{E}^m$, $c(B) \in \mathcal{E}^m$ does not in general imply $\mathcal{E}^n(A) = \mathcal{E}^n(B)$.

§3. $\mathcal{E}^n(A)$ groups and the word problem.

Following the definition given in [1] we say a countable group G is an "$\mathcal{E}^n(A)$ group" (or is "$\mathcal{E}^n(A)$ computable") if it has an "index" (i,m,j) for i an injection of G onto an $\mathcal{E}^n(A)$ decidable subset of N, m an $\mathcal{E}^n(A)$ computable function $m: i(G) \times i(G) \to i(G)$ where m is given by $(i(g_1), i(g_2)) \overset{m}{\mapsto} i(g_1 g_2)$, and likewise for $j: i(G) \to i(G)$ given by $i(g) \overset{j}{\mapsto} i(g^{-1})$. For G_1 and G_2 $\mathcal{E}^n(A)$ computable groups with indices (i_1, m_1, j_1) and (i_2, m_2, j_2) respectively, we say a homomorphism $f: G_1 \to G_2$ is "$\mathcal{E}^n(A)$ computable" if $\hat{f}: i_1(G_1) \to i_2(G_2)$ by $\hat{f}i_1(g) \mapsto i_2 f(G)$ is an $\mathcal{E}^n(A)$ computable function. Note that the computability of f depends on the indices for G_1 and G_2. When not obvious from context, we will say "f is $\mathcal{E}^n(A)$ computable relative to indices (i_1, m_1, j_1) and (i_2, m_2, j_2)".

We freely use the results of §2, the $\mathcal{E}^3$ pairing functions, the $\mathcal{E}^3$ computable functions ##1-21 of Kleene [7] p. 222f (note in particular ##16 through ##21, p. 230), the statements #A through #F of Kleene [7] p. 222f modified by replacing "primitive recursive" by "$\mathcal{E}^3$" and the concept of a group given by generators $a_1, a_2, \cdots$ and relations $R_1, R_2, \cdots$ (see [8]) which will be denoted $G = \langle a_1, \cdots; R_1, \cdots \rangle$.

We begin with a lemma stated as Lemma 4.1 of [1]. In the proof we use a slightly different index which is more convenient to use later.

<u>Lemma 3.1</u> A free group $F = \langle a_1, \cdots ; \rangle$ on finitely or countably many generators is $\mathcal{E}^3$.

<u>Proof:</u> F consists of freely reduced words of the form $w = a_{i_0}^{\alpha_0} \cdots a_{i_r}^{\alpha_r}$ for $\alpha_0, \cdots, \alpha_r$ positive or negative (but non-zero) integers. Write $\bar{\alpha} = 2\alpha$ if $\alpha > 0$ and $\bar{\alpha} = -2\alpha - 1$ if $\alpha < 0$. Then using the pairing functions of §2 we write for $w \neq \Lambda$ (the empty word)

$$i(w) = \prod_{k=0}^{r} p_k \exp J(i_k, \bar{\alpha}_k)$$

$$i(\Lambda) = 1$$

for p_k the k th prime starting with $p_0 = 2, p_1 = 3, \cdots$. Denoting $GN(x) \leftrightarrow \forall k < lh x\left((x)_k \neq 0\right)$ we see that $x \in i(F_n)$ for $F_n = \langle a_0, \cdots, a_{n-1}; \rangle$, finitely generated, iff $x = 1 \vee \left[GN(x) \wedge \forall k < lh x\left(\left(K\left((x)_k\right) < n\right) \wedge \left(L\left((x)_k\right) \neq 0\right)\right)\right]$ and $x \in i(F_\infty)$ for $F_\infty = \langle a_0, \cdots ; \rangle$, countably generated iff $x = 1 \vee \left[GN(x) \wedge \forall k < lh x\left(L\left((x)_k\right) \neq 0\right)\right]$.

To compute $m(x, y)$ one must "decode" x and y as words, freely reduce the concatenation of these words and then encode the result. It is clear that such a process can be interpreted by a recursion defined on $\mathcal{E}^3$ functions. Moreover since $m(x,y) \leq x * y$ (a relation we use later), the recursion is limited and m is an $\mathcal{E}^3$ function. Similarly it is clear that inversion, j, can be performed by a recursion defined on $\mathcal{E}^3$ functions and limited by $j(x) \leq p_{lh x} \exp\left(\left(lh x\right)\left(\max_{0 \leq i < lh x} (x)_i + 1\right)\right)$ so j is $\mathcal{E}^3$ computable. □

Using the above lemma we can show that the study of $\mathcal{E}^n(A)$ groups is non-empty.

<u>Theorem 3.2:</u> For every countable group G there exists $A \subset N$ such that G is an $\mathcal{E}^3(A)$ group.

<u>Proof.</u> G being countable (or finite) there exists a presentation, $1 \to K \to F \to G \to 1$, as an exact sequence, for F free and at most countably generated hence $\mathcal{E}^3$. Let $A = i(K) \subset i(F) \subset N$. Following [1], Theorem 5.1 we define the $\mathcal{E}^3(A)$ predicate

$$E(x,y) \leftrightarrow x \in i(F) \wedge y \in i(F) \wedge m\left(j(x), y\right) \in i(K).$$

Thus the predicate E says "x and y are in the same coset modulo K". We define a unique index for each such coset, and hence for G, by

$$r(x) = \mu y \leq x\left(E(x,y)\right).$$

Now $i_G(G) = ri(F)$ and $x \in i_G(G) \leftrightarrow x \in i(F) \wedge r(x) = x$. We define $m_G(x,y) = rm(x,y)$ and note $m_G(x,y) \leq x * y$. Similarly $j_G(x) = rj(x)$. □

Theorem 3.2 suggests that if G is finitely generated (f.g.), then A is the encoded version of the word problem. The relation between $\mathcal{E}^n(A)$ groups and the word problem is made precise in the following definition, which, for the sake of the discussion further below, will be given for countably generated groups. However, note that the relationship between $\mathcal{E}^n(A)$

groups and the word problem depends on finite systems of generators, only.

Definition 3.3: A group G is said to be $\mathcal{E}^n(A)$ standard relative to an index (i,m,j) if i is defined by minimalization from a presentation $1 \to K \to F \to G \to 1$ for F free on at most countably many generators, $n \geq 3$, and $A \subset N$, the minimalization to be performed on $E(x,y)$ of the proof of Theorem 3.2 for $E(x,y)$ $\mathcal{E}^n(A)$ decidable.

In Definition 3.3 we do not require that $A = i(K)$ but merely that $i(K)$ be $\mathcal{E}^n(A)$ decidable so that $E(x,y)$ is $\mathcal{E}^n(A)$ decidable. Clearly the index given in Theorem 3.2 is a standard index. Intuitively, a finitely generated G is given a standard index by solving the word problem for a presentation of G by an $\mathcal{E}^n(A)$ process. Theorem 3.2 says that the word problem for a countable group can be solved for any given presentation and some A. Our next theorem shows that for finitely generated groups the level of computability of the word problem is independent of the f.g. presentation.

Theorem 3.4: If G is f.g. and $\mathcal{E}^n(A)$ standard for $n \geq 3$ then any standard index of G is $\mathcal{E}^n(A)$.

Proof: Let $1 \to K \to F \xrightarrow{\sigma} G \to 1$ be a presentation of G for F finitely generated. We show K is $\mathcal{E}^n(A)$ decidable by showing σ is $\mathcal{E}^n(A)$ computable relative to the index (i,m,j) on F and (i',m',j'), the given $\mathcal{E}^n(A)$ standard index of G. Then $x \in i(K) \leftrightarrow x \in i(F) \wedge \hat{\sigma}(x) = i'(1)$.

For $w \in F$, $\sigma(w)$ is computed as the product of images of generators of F corresponding to the spelling of w. Thus since F is finitely generated, $\hat{\sigma}(x)$ can be interpreted by a recursion on $\mathrm{lh}(x)$ involving m' where $\hat{\sigma}$ is $\mathcal{E}^n(A)$ computable if the recursion is bounded. Here $m'(y,z) \leq y * z$ since (i',m',j') is a standard index so $\hat{\sigma}(x)$ is bounded by $y_1 * y_2 * \cdots * y_r$ where each y_i as well as r can be computed from x by an $\mathcal{E}^3$ function. But $\mathop{*}\limits_{i=1}^{r} y_i = y_1 * \cdots * y_r \leq p_k \exp\left(k \sum_{i=1}^{r} \sum_{n=0}^{\mathrm{lh} y_i \dot{-} 1} (y_i)_n\right)$ for $k = \sum_{i=1}^{r} \mathrm{lh}\, y_i$. Thus the recursion is bounded and $\hat{\sigma}$ is $\mathcal{E}^n(A)$ computable. □

Corollary 3.5: If G is f.g. and $\mathcal{E}^n(A)$ computable for $n \geq 3$ then G is $\mathcal{E}^{n+1}(A)$ standard.

Proof: In the proof of Theorem 3.4 if G is $\mathcal{E}^n(A)$ but not $\mathcal{E}^n(A)$ standard, the recursion defining $\hat{\sigma}$ need not be bounded and therefore $\hat{\sigma}$ and hence $i(K)$ may be $\mathcal{E}^{n+1}(A)$ rather than $\mathcal{E}^n(A)$ by Theorem 2.4. □

Corollary 3.6: If G is f.g. and $\mathcal{E}^n(A)$ computable for $n \geq 3$ then G is $\mathcal{E}^3(B)$ standard for B $\mathcal{E}^{n+1}(A)$ decidable.

Proof: In the case of Corollary 3.5, set $B = i(K)$. Then B is $\mathcal{E}^{n+1}(A)$ decidable and G is $\mathcal{E}^3(B)$ standard by the usual construction of Theorem 3.2. □

Corollary 3.7: If G is an $\mathcal{E}^n(A)$ computable group for $n \geq 3$ and $H < G$

is a finitely generated subgroup then H is $\mathcal{E}^{n+1}(A)$ standard. If G is $\mathcal{E}^{n}(A)$ standard then H is $\mathcal{E}^{n}(A)$ standard.

Proof: In the proof of Theorem 3.4 and Corollary 3.5 we did not need σ surjective. Corollary 3.7 is then a restatement for σ not surjective and $H \cong F/K$. □

Corollary 3.8: If G is f.g. and $\mathcal{E}^{n}$ standard but not $\mathcal{E}^{n-1}$ standard for $n \geq 4$, then G is not $\mathcal{E}^{m}$ for $m < n-1$.

Proof: This is a restatement of Corollary 3.5 for $A = N$. □

Theorem 3.4 above is a mild generalization of a result due to Rabin, [9], that the computability of the word problem depends on G and not any of G's finitely generated presentations. Corollaries 3.5, 3.6, and 3.8 show the relationship between standard and non-standard indices, Corollary 3.8 is a special case for later use. Corollary 3.7 shows that the property of being $\mathcal{E}^{n}(A)$ standard is inherited by finitely generated subgroups although in general a finitely generated subgroup of an $\mathcal{E}^{n}(A)$ group may be $\mathcal{E}^{n+1}(A)$. From the proof of Corollary 3.7 we also see that the embedding $H \to G$ is $\mathcal{E}^{n+1}(A)$ computable since it can be computed by regarding an element in the index of H to be in $i(F)$ and applying the $\mathcal{E}^{n+1}(A)$ computable $\hat{\sigma}$. As a companion to the hereditary result of Corollary 3.7 we see that under suitable conditions, quotient groups of $\mathcal{E}^{n}(A)$ groups are $\mathcal{E}^{n}(A)$.

Proposition 3.9. If G is an $\mathcal{E}^{n}(A)$ group for $n \geq 3$ and K G is an $\mathcal{E}^{n}(A)$ decidable normal subgroup then G/K is $\mathcal{E}^{n}(A)$ computable.

Proof: In the proof of Theorem 3.2 replace F by G and G by G/K and let (i,m,j) be the original index for G. The same definitions of E, c, m and j work as does the decidability criteria. □

It is of course not true that all quotient groups of $\mathcal{E}^{n}(A)$ are $\mathcal{E}^{n}(A)$ for if that were the case all groups would be $\mathcal{E}^{3}$ and, in particular, have a solvable word problem. Note however that if $B = i(K) \subset i(G)$ in Proposition 3.9 then by replacing A by

$$A \text{ join } B = \left\{ x \,\middle|\, 2 \mid x \Rightarrow [x/2] \in A \;\&\; 2 \nmid x \Rightarrow [(x \dot{-} 1)/2] \in B \right\}$$

we see G/K is $\mathcal{E}^{n}(A \text{ join } B)$ computable.

§4. Free products with amalgamation.

The free product with amalgamation is a useful construction when dealing with decision problems in groups since intuitively the normal form theorem yields decision procedures for such products modulo the decision procedures for the groups and the amalgamated subgroups. In the following we will study the free product with amalgamation for $\mathcal{S}^n(A)$ groups, $n \geq 3$, where the subgroups amalgamated are themselves $\mathcal{S}^n(A)$ as is the isomorphism relating them. In this context recall that if one decision problem has an $\mathcal{S}^p(B)$ solution and another an $\mathcal{S}^q(C)$ solution then they both have $\mathcal{S}^n(A)$ solutions for $n = \max(p,q)$ and $A = B$ join $C = \left\{x \,\middle|\, 2|x \Rightarrow [x/2] \in A \;\&\; 2\nmid x \Rightarrow [x \dot{-} 1/2] \in B\right\}$. In view of this our hypothesis will always involve a single computability class, $\mathcal{S}^n(A)$.

In general we require not only the index of the product but also the manner in which the original factors are embedded. The following definition is used to make such statements precise.

<u>Definition 4.1</u>: Let G and L be $\mathcal{S}^n(A)$ groups with indices (i,m,j) and (i',m',j') respectively. Suppose $\kappa : G \to L$ is an embedding. Then we say κ is an <u>$\mathcal{S}^n(A)$ embedding of G into L</u> (or simply G is <u>$\mathcal{S}^n(A)$ embedded</u> into L) if:

(i) $\hat{\kappa} : i(G) \to i'(L)$ is $\mathcal{S}^n(A)$ computable with respect to (i,m,j) and (i',m',j'), (i.e. κ is $\mathcal{S}^n(A)$ computable).

(ii) $\hat{\kappa} i(G) \subset i'(L)$ is $\mathcal{S}^n(A)$ decidable, (i.e., G is an $\mathcal{S}^n(A)$ decidable subgroup of L).

(iii) $\hat{\kappa}^{-1} : \hat{\kappa} i(G) \to i(G)$ is $\mathcal{S}^n(A)$ computable with respect to (i',m',j') and (i,m,j), (i.e., κ^{-1} is $\mathcal{S}^n(A)$ computable).

Notice that Definition 4.1 involves the particular choice of indices being used. In the following, the particular choice of indices is specified only when not obvious from context. Also observe that in the case where G is f.g. and (i,m,j) and (i',m',j') are standard indices, conditions (i) and (iii) are superfulous.

We demonstrate the type of result we desire for free products with amalgamation by first considering the more obvious situation for direct product, denoted $G_1 \times G_2$.

<u>Proposition 4.2:</u> Let G_a be $\mathcal{S}^n(A)$ groups for $n \geq 3$ and $a = 1,2$. Then $G = G_1 \times G_2$ is an $\mathcal{S}^n(A)$ group and the embeddings $G_a \to G$ are $\mathcal{S}^n(A)$ embeddings.

<u>Proof</u>: Let G_a have indices (i_a, m_a, j_a) and give G an index by $i(g_1, g_2) = J(i_1(g_1), i_2(g_2))$. It is clear this index has all of the prescribed properties. □

We may ask about the universal properties of the direct product. Clearly the projections are $\mathcal{S}^n(A)$ computable.

<u>Proposition 4.3:</u> Under the assumptions of Proposition 4.2 assume also K is an $\mathcal{E}^n(A)$ group and $\alpha_a : K \to G_a$ are $\mathcal{E}^n(A)$ computable homomorphisms. Let $\pi_a : G \to G_a$ be the projections and $\alpha : K \to G$ the unique homomorphism such that $\pi_a \alpha = \alpha_a$. Then α is $\mathcal{E}^n(A)$ computable.

<u>Proof:</u> $\hat{\alpha}(x) = J\left(\hat{\alpha}_1(x), \hat{\alpha}_2(x)\right)$. □

As we saw the index given $G = G_1 \times G_2$ by Proposition 4.2 was the natural index with regard to the universal property. However, if G_1, G_2 (and hence G) are f.g. then it is not the standard index. Next we relate this index to the standard index.

<u>Proposition 4.4:</u> Under the assumptions of Proposition 4.2 assume also that G_a are f.g. and $\mathcal{E}^n(A)$ standard. Then G is $\mathcal{E}^n(A)$ standard and the identity isomorphism on G is $\mathcal{E}^n(A)$ computable from G with standard index to G with index (i,m,j) of Proposition 4.2 and from G with index (i,m,j) to G with standard index.

<u>Proof</u>: Let G_1 be generated by $a_1, \cdots, a_r$ and G_2 by $a_{r+1}, \cdots, a_s$. Consider $F = \langle a_1, \cdots, a_{r+s}; \rangle$ and $\alpha_1 : F \to G_1$ by $a_i \mapsto a_i$ for $i \le r$, $a_i \mapsto 1$ for $i > r$, and $\alpha_2 : F \to G_2$ by $a_i \mapsto 1$ for $i \le r$, $a_i \mapsto a_i$ for $i > r$. Then G_a being $\mathcal{E}^n(A)$ standard the α_a are $\mathcal{E}^n(A)$ computable and induce $\alpha : F \to G$ which is $\mathcal{E}^n(A)$ computable by Proposition 4.3. Thus $\ker \alpha$ is $\mathcal{E}^n(A)$ decidable so G is $\mathcal{E}^n(A)$ standard. This argument also shows that the identity isomorphism on G is $\mathcal{E}^n(A)$ computable from the standard index to the index of Proposition 4.2. Conversely since the G_a have standard indices it is an $\mathcal{E}^n(A)$ process to write $(g_1, g_2) = \left(w_1(a_i), w_2(a_i)\right)$ as words on the generators (w_1 involves only $a_1, \cdots, a_r$ and w_2 only $a_{r+1}, \cdots, a_{r+s}$) and so associate (g_1, g_2) with $w = w_1(a_i) w_2(a_i) \in F$. Reducing w modulo $\ker \alpha$, the identity isomorphism on G is $\mathcal{E}^n(A)$ computable from the index of Proposition 4.2 to the standard index. □

We will proceed along the same lines for free product with amalgamation. That is, first we will develop an index which is natural, then verify the universal property and finally, restricting our attention to f.g. groups show the relationship to the standard index. In this case the natural index will reflect the normal form representation of elements in the free product. Informally this index will be called the normal form index. Since the normal form requires coset representatives in the factors modulo the amalgamated subgroup (amalgam for short) we use the following definition (compare [4] Definition 2.2).

<u>Definition 4.5:</u> Let G be an $\mathcal{E}^n(A)$ group and $H < G$ an $\mathcal{E}^n(A)$ decidable subgroup for $n \ge 3$. An <u>$\mathcal{E}^n(A)$ right coset representative system</u> for G mod H is an $\mathcal{E}^n(A)$ computable function $k : i(G) \to i(G)$ satisfying:

(i) $x \in i(G) \wedge y \in i(G) \wedge m\left(x, j(y)\right) \in i(H) \to k(x) = k(y)$

(ii) $m\left(x, jk(x)\right) \in i(H)$

(iii) $x \in i(H) \to k(x) = i(1)$.

Intuitively k is a method of choosing right coset representatives for elements of G by an $\mathcal{E}^n(A)$ process. There always exists $\mathcal{E}^n(A)$ right coset representative systems for example for $x \in i(G)$ define $k(x) = \mu y < x\big(y \in i(G) \wedge m\big(x, j(y)\big) \in i(H)\big)$ if $x \notin i(H)$ and $k(x) = i(1)$ if $x \in i(H)$. Notice that any k as in the definition decomposes $x \in i(H)$ as $x = m\big(h(x), k(x)\big)$ for $h(x) = m\big(x, jk(x)\big)$, i.e. allows us to write $g = hg'$ for $h \in H$ and g' a particular coset representative by an $\mathcal{E}^n(A)$ process. Our use of such representative systems is seen in the following.

<u>Theorem 4.6:</u> Let G_a be $\mathcal{E}^n(A)$ groups for $n \geq 3$ and $a = 1,2$. Assume $H_a < G_a$ are $\mathcal{E}^n(A)$ decidable subgroups and $\varphi : H_1 \to H_2$ is an isomorphism such that φ and φ^{-1} are $\mathcal{E}^n(A)$ computable. Then for each choice of $\mathcal{E}^n(A)$ right coset representative systems $k_a : i_a(G_a) \to i_a(G_a)$ there is an $\mathcal{E}^{n+1}(A)$ index on $G_1 *_\varphi G_2$ (the free product of G_1 and G_2 amalgamating H_1 and H_2 by φ). The natural embeddings $G_a \to G$ are $\mathcal{E}^{n+1}(A)$ embeddings.

<u>Proof:</u> This is a relativized version of Theorem 2.1. of [4] replacing the k_a of that proof by the given k_a. □

<u>Corollary 4.7:</u> Let G_a be $\mathcal{E}^n(A)$ groups for $n \geq 3$ and $a = 1,2$. Then $G_1 * G_2$ is an $\mathcal{E}^n(A)$ group and the natural embeddings $G_a \to G_1 * G_2$ are $\mathcal{E}^n(A)$ embeddings.

<u>Proof:</u> Relativize Corollary 2.1.1 of [4]. □

We now have the universal property.

<u>Corollary 4.8:</u> Under the assumptions of Theorem 4.6 assume also that is an $\mathcal{E}^n(A)$ or an $\mathcal{E}^{n+1}(A)$ standard group and $\tau_a : G_a \to K$ are $\mathcal{E}^n(A)$ computable homomorphisms agreeing on the amalgam (i.e., if $h \in H_1$ then $\tau_1(h) = \tau_2\varphi(h)$). Then the unique homomorphism $\tau : G_1 *_\varphi G_2 \to K$ extending τ_1 and τ_2 (i.e., such that $\tau|G_1 = \tau_1$ and $\tau|G_2 = \tau_2$) is $\mathcal{E}^{n+1}(A)$ computable.

<u>Proof:</u> Relativize Corollary 2.1.2 of [4] for K $\mathcal{E}^n(A)$. For K $\mathcal{E}^{n+1}(A)$ standard the bound on multiplication in K yields a bound on the recursion for $\bar{\tau}$. □

We now can relate indices of $G_1 *_\varphi G_2$ which arise from different choices of the coset representative systems (compare [4] Proposition 2.2).

<u>Corollary 4.9:</u> Under the assumptions of Theorem 4.6 let $G_1 *_\varphi G_2$ have index (i,m,j) with respect to $\mathcal{E}^n(A)$ coset representative systems k_a and (i',m',j') with respect to $\mathcal{E}^n(A)$ coset representative systems k'_a. Then the identity isomorphism on $G_1 *_\varphi G_2$ is $\mathcal{E}^{n+2}(A)$ computable relative to (i,m,j) and (i',m',j').

<u>Proof:</u> The embeddings $G_a \to G_1 *_\varphi G_2$ are $\mathcal{E}^n(A)$ computable with respect to (i_a, m_a, j_a) and (i',m',j') so extend to the identity isomorphism on $G_1 *_\varphi G_2$, $\mathcal{E}^{n+2}(A)$ computable with respect to (i,m,j) and (i',m',j') by Corollary 4.8 with $K = G_1 *_\varphi G_2$, $\mathcal{E}^{n+1}(A)$ computable. □

We now restrict our attention to f.g. groups and consider standard indices.

<u>Theorem 4.10</u>: Let G_a be f.g. $\mathcal{E}^n(A)$ standard groups for $n \geq 3$ and $a = 1,2$. Assume $H_a < G_a$ are $\mathcal{E}^n(A)$ decidable subgroups and $\varphi: H_1 \to H_2$ is an isomorphism such that φ and φ^{-1} are $\mathcal{E}^n(A)$ computable. Then $G_1 *_\varphi G_2$ is $\mathcal{E}^{n+1}(A)$ standard.

<u>Proof</u>: We show that the word problem is $\mathcal{E}^{n+1}(A)$ decidable for a particular f.g. presentation of $G_1 *_\varphi G_2$. The argument will be a "spelling" argument but it should be clear that it can be encoded as a recursion defined on $\mathcal{E}^n(A)$ functions and hence $G_1 *_\varphi G_2$ is $\mathcal{E}^{n+1}(A)$ standard.

Let G_1 be generated by $a_1, \cdots, a_r$ and G_2 by $a_{r+1}, \cdots, a_{r+s}$. Since G_a are $\mathcal{E}^n(A)$ standard there is an $\mathcal{E}^n(A)$ process for recognizing if a word on the $a_1, \cdots, a_r$ is in H_1 and if so computing a word on $a_{r+1}, \cdots, a_{r+s}$ corresponding to its image under φ. Similarly one can compute φ^{-1} of a word in H_2. The statement that these processes are $\mathcal{E}^n(A)$ requires that the original indices of the G_a be standard.

Let w be any freely reduced word on the a_i. If the first symbol in w is a power of a_k for $1 \leq k \leq r$, write $w = w_1 \cdots w_p$ for each $w_j \neq \emptyset$ and so that j odd implies w_j involves only symbols a_k for $1 \leq k \leq r$ and j even implies w_j involves only symbols a_k for $r+1 \leq k \leq r+s$. Similarly if the first symbol is a power of a_k for $r+1 \leq k \leq r+s$ write $w = w_1 \cdots w_p$ as above interchanging the roles of even and odd. In [8] such w_j are

called syllables and p is called the syllable length of w. Clearly the decomposition of w into syllables is an $\mathcal{E}^3$ process.

We proceed by induction on the syllable length p of w. If $p = 1$ then $w = 1$ in $G_1 *_\varphi G_2$ iff $w = 1$ in G_1 or $w = 1$ in G_2, an $\mathcal{E}^n(A)$ decision. If $p > 1$ let $1 \leq q \leq p$ be the smallest integer such that either $w_q \in H_1$ or $w_q \in H_2$. If there is no such q, $w \neq 1$ in $G_1 *_\varphi G_2$. The search for q can be interpreted as bounded minimalization defined by the $\mathcal{E}^3$ function which decomposes w and the $\mathcal{E}^n(A)$ decision functions for H_a. If there is a q and $w_q \in H_1$, apply φ to w_q as described above and replace w by w' such that $w = w'$ in $G_1 *_\varphi G_2$ but w' has shorter syllable length than w. Similarly if $w_q \in H_2$ apply φ^{-1} to get w'. By induction we conclude that the word problem for $G_1 *_\varphi G_2$ is $\mathcal{E}^{n+1}(A)$ solvable.

Thus if $B \subset N$ is the encoded image of the kernel in the presentation of $G_1 *_\varphi G_2$ on $a_1, \cdots, a_{r+s}$, then $G_1 *_\varphi G_2$ is $\mathcal{E}^3(B)$ computable by Theorem 3.2 and $\mathcal{E}^{n+1}(A)$ computable since B is $\mathcal{E}^{n+1}(A)$ decidable by the above. □

<u>Corollary 4.11</u>: Let G_a be f.g. $\mathcal{E}^n(A)$ standard groups for $n \geq 3$ and $a = 1,2$. Then $G_1 * G_2$ is $\mathcal{E}^n(A)$ standard.

<u>Proof</u>: In the proof of Theorem 4.10, the questions $w_k \in H_a$ are replaced by $w_k = 1$ in G_1 or G_2. Since the index of w' (and hence every succeeding reduction in the induction) is less than the index of w (there is no φ or φ^{-1} to possibly increase the index), the recursion used in an encoded

version of the proof of Theorem 4.10 is bounded and the word problem $\mathcal{E}^n(A)$ decidable. □

We now study the relationship between the "normal form" index of Theorem 4.6 and the standard index of Theorem 4.10.

Corollary 4.12: Under the assumptions of Theorem 4.10 let (i',m',j') be the $\mathcal{E}^{n+1}(A)$ standard index and (i,m,j) be the $\mathcal{E}^{n+1}(A)$ (normal form) index given by Theorem 4.6 arising from some $\mathcal{E}^n(A)$ right coset representative system. Then the identity isomorphism on $G_1 *_\varphi G_2$ is $\mathcal{E}^{n+1}(A)$ computable from the index (i,m,j) to the index (i',m',j'). It is $\mathcal{E}^{n+2}(A)$ computable from the index (i',m',j') to the index (i,m,j).

Proof: Since G_a are f.g. and $\mathcal{E}^n(A)$ standard, there are $\mathcal{E}^n(A)$ quotient maps $\zeta_a : F^{(a)} \to G_a$ for $F^{(a)}$ f.g. free groups. Moreover for $x \in i_a(G_a)$, $x \in i_a(F^{(a)})$ with $\hat{\sigma}_r(x) = x$. Let $\sigma : F \to G_1 *_\varphi G_2$ be the $\mathcal{E}^{n+1}(A)$ quotient map for $F = F^{(1)} * F^{(2)}$ and $\tau_a : F^{(a)} \to F$ the $\mathcal{E}^3$ embeddings. Then the embedding $G_a \to G_1 *_\varphi G_2$ relative to (i_a, m_a, j_a) and (i',m',j') is given by $x \mapsto \hat{\sigma}\hat{\tau}(x)$ and so is $\mathcal{E}^{n+1}(A)$ computable. By Corollary 4.8 the extension of the embeddings, which is obviously the identity, is $\mathcal{E}^{n+1}(A)$ computable, the K of Corollary 4.8 being $G_1 *_\varphi G_2$ and having a standard index. The second statement of the Corollary is immediate from the fact that the quotient $\sigma : F \to G_1 *_\varphi G_2$ is $\mathcal{E}^{n+2}(A)$ computable relative to the index (i,m,j) (since F is f.g.). □

Corollary 4.13: Under the assumptions of Corollary 4.11 let (i',m',j') be the $\mathcal{E}^n(A)$ standard index and let (i,m,j) be the $\mathcal{E}^n(A)$ (normal form) index given by Corollary 4.7. Then the identity isomorphism on $G_1 * G_2$ is $\mathcal{E}^n(A)$ computable from the index (i,m,j) to (i',m',j') and also from the index (i',m',j') to the index (i,m,j).

Proof: In the proof of Corollary 4.12 the quotient map σ is $\mathcal{E}^n(A)$ computable when the amalgam is trivial and so by the remainder of the argument the identity is $\mathcal{E}^n(A)$ from the index (i,m,j) to (i',m',j'). The technique used to bound the index of a product in the proof of Corollary 4.7 ([4] Corollary 2.1.1) can be used to bound the recursion used in the computability of $\sigma : F \to G_1 * G_2$ relative to the index (i,m,j). Then σ is $\mathcal{E}^n(A)$ computable and hence so is the identity. □

By a standard technique Theorem 4.10 can be used to construct f.g. $\mathcal{E}^3(A)$ groups for any $A \subset N$ in the following sense.

Corollary 4.14: For any $A \subset N$ there exists a f.g. $\mathcal{E}^3(A)$ standard group G_A such that if G_A is $\mathcal{E}^n(B)$ standard for $n \geq 3$ then A is $\mathcal{E}^n(B)$ decidable.

Proof: Set $F' = \langle a,b; \rangle$, $F'' = \langle c,d; \rangle$, $H' < F'$ the subgroup generated by $\{a^x b a^{-x}; x \in A\}$ and $H'' < F''$ the subgroup generated by $\{c^x d c^{-x}; x \in A\}$. Then H' is freely generated by the $a^x b a^{-x}$ for $x \in A$ and so is $\mathcal{E}^3(A)$ decidable. Similarly for H''. Let $\varphi : H' \to H''$ be the $\mathcal{E}^3(A)$ computable

isomorphism given by $a^x b a^{-x} \mapsto c^x d c^{-x}$ for $x \in A$. We consider $G_A = F' *_\varphi F''$. By Theorem 4.10 G_A is $\mathcal{E}^4(A)$ computable but since $\hat{\varphi}$ and $\hat{\varphi}^{-1}$ are restrictions of the identity on N, the index of each successive w' (in the proof of Theorem 4.10) is less than or equal to $p_{lhx} \exp\left(J\left(4 \max_{0 \le q < lhx} L((x)_q)\right) \cdot lhx\right)$ for x the index of w and hence the recursion used in solving the word problem is bounded so G_A is $\mathcal{E}^3(A)$ standard.

Suppose G_A is $\mathcal{E}^n(B)$ standard for $n \geq 3$. Then for any $x \in N$, to decide if $x \in A$, compute the index of $a^x b a^{-x} c^x d^{-1} c^{-x}$ in G_A by an $\mathcal{E}^n(B)$ process. This process is a deicison procedure for A since $x \in A$ iff $a^x b a^{-x} c^x d^{-1} c^{-x} = 1$ in G_A. □

The above constructions for f.g. $\mathcal{E}^n(A)$ groups yield interesting corollaries for $\mathcal{E}^n = \mathcal{E}^n(N)$ groups. In particular we consider $\mathcal{E}^3(A)$ groups for A $\mathcal{E}^n$ decidable with $n > 3$.

<u>Proposition 4.15:</u> Under the assumptions of Theorem 4.10 let $n = 3$ and A be $\mathcal{E}^n$ decidable for $n \geq 4$. Then $G_1 *_\varphi G_2$ has an $\mathcal{E}^n$ index for each $\mathcal{E}^3(A)$ coset representative system. Moreover $G_1 *_\varphi G_2$ is $\mathcal{E}^n$ standard and for $n \geq 5$ the identity isomorphism is $\mathcal{E}^n$ computable with respect to any of the above indices.

<u>Proof:</u> Since $n \geq 4$ and A is $\mathcal{E}^n$ decidable $\mathcal{E}^{3+1}(A) \subset \mathcal{E}^n$ so the first part of the statement is immediate from Theorems 4.6 and 4.10. Similarly $\mathcal{E}^{3+2}(A) \subset \mathcal{E}^n$ for $n \geq 5$ so the second part follows from Corollaries 4.9 and 4.12. □

Of course we could specialize other statements to the case of $\mathcal{E}^n$ groups, e.g. Corollary 4.8. Of special interest is the following specialization of Corollary 4.14.

<u>Proposition 4.16:</u> For $n \geq 4$ there exist $\mathcal{E}^n$ standard groups which are not $\mathcal{E}^{n-1}$ standard. For $n \geq 5$ there exist $\mathcal{E}^n$ standard groups which are not $\mathcal{E}^{n-2}$.

<u>Proof:</u> In [1] it is shown that for $n \geq 4$ there exists sets $\mathcal{E}^n$ decidable but not $\mathcal{E}^{n-1}$ decidable. If A is such a set then the group G_A of Corollary 4.14 is $\mathcal{E}^3(A) \subset \mathcal{E}^n$ standard but not $\mathcal{E}^{n-1}$ standard since that would imply A $\mathcal{E}^{n-1}$ decidable. If $n \geq 5$ then G_A is not $\mathcal{E}^{n-2}$ since by Corollary 3.8 that would imply G_A is $\mathcal{E}^{n-1}$ standard. □

We say a group is primitive recursive (p.r.) if it is $\mathcal{E}^n$ for some n. We say a group is finitely presented (f.p.) if it has a f.g. presentation involving only finitely many relations. C. W. Miller III raised the question of whether there could exist a p.r., f.p. group containing as subgroups a copy of every p.r., f.p. group. As reported in [2] the answer is strongly negative.

<u>Proposition 4.17:</u> There does not exist a p.r. group containing every f.g., p.r. group as a subgroup. There does not exist a p.r. group containing every f.p., p.r. group as a subgroup.

<u>Proof</u> By Proposition 4.16 there exist f.g. $\mathcal{S}^n$ groups not $\mathcal{S}^{n-2}$ for $n \geq 5$. Applying Theorem 5.1 of [4] there exist f.p. $\mathcal{S}^m$ groups for $m \geq n-2$ and every $n \geq 5$. Now suppose there was a p.r. group G containing a copy of every p.r., f.g. (respectively f.p.) group as a subgroup. Then G is $\mathcal{S}^p$ for some p so by Corollary 3.7 each of its f.g. subgroups is $\mathcal{S}^{p+1}$ which is a contradiction since we have just seen there exist f.g. (respectively f.p.) $\mathcal{S}^m$ groups for arbitrarily high m. □

The remainder of this section is devoted to a technical devise for finding the computability of certain special subgroups of a free product with amalgamation. The computability of these subgroups is of importance for some later applications so the statements are included here for completeness. The casual reader may choose to ignore this material. We present a generalization of Definition 2.3, Proposition 2.3 and Lemma 2.1 of [4].

<u>Definition 4.18</u>: Let G be an $\mathcal{S}^n(A)$ group for $n \geq 3$ and $H<G$ an $\mathcal{S}^n(A)$ decidable subgroup. A subgroup $K<G$ is said to be <u>$H, \mathcal{S}^n(A)$ compatible</u> if it is $\mathcal{S}^n(A)$ decidable and there exists an $\mathcal{S}^n(A)$ right coset representative system for G mod H satisfying in addition to conditions (i) through (iii) of Definition 4.5:

(iv) $\quad x \in i(K) \rightarrow k(x) \in i(K)$.

What we require in Definition 4.18 is the existence of an $\mathcal{S}^n(A)$ right coset representative system for G mod H so that whenever a coset Hg satisfies $Hg \cap K \neq \emptyset$ then the representative of Hg is in K. Notice

condition (iv) is trivially satisfied when $H<K<G$, all $\mathcal{S}^n(A)$ decidable, for then $g \in K, \hat{g} \in G, g\hat{g}^{-1} \in H$ implies $g\hat{g}^{-1} \in K$ so $\hat{g} \in K$ and any coset representative of Hg is in K. The following lemma characterizes compatibility and is more convenient than the definition for applications.

<u>Lemma 4.19</u>: Let G be $\mathcal{S}^n(A)$ with index (i,m,j) such that $0 \notin i(G)$. Let $H<G$, $K<G$ be $\mathcal{S}^n(A)$ decidable subgroups. The following are equivalent:

(1) there exists an $\mathcal{S}^n(A)$ computable function $d: i(G) \rightarrow i(K) \cup \{0\}$ such that $d(x)=0 \leftrightarrow \neg\exists y\big(y \in i(K) \wedge m(x,j(y)) \in i(H)\big)$ and $d(x) \neq 0 \rightarrow m(x, jd(x)) \in i(H)$

(2) K is $H, \mathcal{S}^n(A)$ compatible

(3) H is $K, \mathcal{S}^n(A)$ compatible.

<u>Proof</u>: Replace every occurance of "p.r." in the proof of Proposition 2.3 of [4] by "$\mathcal{S}^n(A)$". □

Observe that the condition $0 \notin i(G)$ in Lemma 4.19 is not critical since if $0 \in i(G)$, $i(g)$ can be replaced by $i(g)+1$. Assume $H<G$, $K<G$ all $\mathcal{S}^n(A)$ and also that $\{H,K\}<G$, the subgroup of G generated by H and K, is $\mathcal{S}^n(A)$ decidable. Then we may take $d(x)=0$ for $x \notin i(\{H,K\})$ since if for $g \in G$ there exists $\hat{g} \in K$ such that $g\hat{g}^{-1}=h \in H$ then $g=h\hat{g} \in \{H,K\}$. Thus to conclude K is $H, \mathcal{S}^n(A)$ compatible in G it suffices to show K is $H, \mathcal{S}^n(A)$ compatible in $\{H,K\}$. The use of the

notion of compatibility is found in the following lemma.

Lemma 4.20. Under the conditions of Theorem 4.6 let $K_a < G_a$ be $\mathcal{E}^n(A)$ decidable subgroups satisfying

(i) K_a is H_a, $\mathcal{E}^n(A)$ compatible for $a = 1,2$

(ii) $\varphi(K_1 \cap H_1) = K_2 \cap H_2$.

Then $K = \{K_1, K_2\} < G = G_1 *_\varphi G_2$ satisfies

(1) K is $\mathcal{E}^{n+1}(A)$ decidable in G with respect to some normal form index on G hence $\mathcal{E}^{n+2}(A)$ decidable in G with respect to any normal form index or any standard index on G (i.e. the embedding $K < G$ is an $\mathcal{E}^{n+2}(A)$ embedding)

(2) for $\varphi' = \varphi|K_1 \cap H_1$, $K = K_1 *_{\varphi'} K_2$

(3) $G_a \cap K = K_a$ for $a = 1,2$.

Proof. Since by Corollaries 4.9 and 4.12 all normal form indices and all standard indices are related by an $\mathcal{E}^{n+2}(A)$ computable identity isomorphism on G it suffices to show K is $\mathcal{E}^{n+1}(A)$ decidable relative to the particular $\mathcal{E}^n(A)$ right coset representative systems with respect to which the K_a are H_a $\mathcal{E}^n(A)$ compatible. To show this as well as (2) and (3) replace every occurance of "p.r." in the proof of Lemma 2.1 of [4] by "$\mathcal{E}^{n+1}(A)$" except that "p.r. right coset representative system" should be replaced by "$\mathcal{E}^n(A)$ right coset representative system". □

§5. Strong Britton extensions

In this section we consider a construction closely related to the free product with amalgamation, the so called strong Britton extension. We will refer to presentations of groups on generators and relations, viz. $G = \langle a_1, \cdots; R_1, \cdots \rangle$ and, when the meaning is clear, interpret this notation with liberty. For example, we may write $\langle G, r; \rangle$ to mean $\langle a_1, \cdots, r; R_1, \cdots \rangle = G * \langle r; \rangle$; i.e. when the set of generators has some implied relations, we assume they are included in the relations of the presentation. We also use the notation $\{B\} < G$ for $B \subset G$, $\{B\}$ the subgroup of G generated by B. The following definitions are relativized versions of Definitions 3.1 and 3.2 of [4].

Definition 5.1: Let G be an $\mathcal{E}^n(A)$ group. An $\mathcal{E}^n(A)$ isomorphism in G is an isomorphism $\varphi: H \to H'$ such that H, H' are $\mathcal{E}^n(A)$ decidable subgroups of G and φ, φ^{-1} are $\mathcal{E}^n(A)$ computable (relative to the inherited index on H, H').

Definition 5.2: Let G be an $\mathcal{E}^n(A)$ group and φ an $\mathcal{E}^n(A)$ isomorphism in G. The strong Britton extension of G by φ, denoted by G_φ, is the group with presentation $\langle G, t; tht^{-1} = \varphi(h) \ \forall h \in \text{domain } \varphi \rangle$.

The $\mathcal{E}^n(A)$ computability of G and φ induces a computability structure in G_φ according to the following theorem and corollary.

Theorem 5.3: Let G be an $\mathcal{E}^n(A)$ group and φ an $\mathcal{E}^n(A)$ isomorphism in G, for $n \geq 3$. Then G_φ is an $\mathcal{E}^{n+2}(A)$ group with index (i', m', j') inherited as a subgroup of a (particular) free product with amalgamation and its

associated normal form index. The embedding $G \to G_\varphi$ is an $\mathcal{S}^{n+2}(A)$ embedding relative to (i',m',j').

Proof: This is a relativized version of Theorem 3.1 of [4]. The proof is identical replacing $\mathcal{S}^n$ by $\mathcal{S}^n(A)$. □

Corollary 5.4: If G is f.g. and $\mathcal{S}^n(A)$ standard in the hypothesis of Theorem 5.3, then G_φ is $\mathcal{S}^{n+1}(A)$ with index (i',m',j') and $G \to G_\varphi$ is an $\mathcal{S}^{n+1}(A)$ embedding. In addition, G_φ is $\mathcal{S}^{n+1}(A)$ standard with index $(\hat{i},\hat{m},\hat{j})$. The identity isomorphism on G_φ is $\mathcal{S}^{n+1}(A)$ computable from index (i',m',j') to index $(\hat{i},\hat{m},\hat{j})$ and $\mathcal{S}^{n+2}(A)$ computable from index $(\hat{i},\hat{m},\hat{j})$ to index (i',m',j'). With respect to $(\hat{i},\hat{m},\hat{j})$ the embedding $G \to G_\varphi$ is $\mathcal{S}^{n+1}(A)$ computable and an $\mathcal{S}^{n+2}(A)$ embedding.

Proof: We modify the proof of Theorem 3.1 of [4]. Since G is f.g. and $\mathcal{S}^n(A)$ standard, L has an $\mathcal{S}^{n+1}(A)$ standard index, say $(\tilde{i},\tilde{m},\tilde{j})$, by Theorem 4.10. We compute $\tau: L \to L$ from index (i',m',j') to index $(\tilde{i},\tilde{m},\tilde{j})$, where the latter index is an $\mathcal{S}^{n+1}(A)$ standard index. Corollary 4.6 shows τ_1 induced by τ_{11} and τ_{12} is automatically $\mathcal{S}^{n+1}(A)$ computable as is τ_2 induced by τ_{21} and τ_{22}. Similarly τ induced by τ_1 and τ_2 is $\mathcal{S}^{n+1}(A)$ computable. Thus $G_\varphi < L$ is $\mathcal{S}^{n+1}(A)$ decidable. Since the embedding $G \to L$ is an $\mathcal{S}^{n+1}(A)$ embedding and $G < G_\varphi$, this shows $G \to G_\varphi$ is an $\mathcal{S}^{n+1}(A)$ embedding.

To see G_φ is $\mathcal{S}^{n+1}(A)$ standard, observe that it is a f.g. subgroup of the $\mathcal{S}^{n+1}(A)$ standard group L and so Corollary 3.7 applies.

To see that the identity isomorphism on G_φ is $\mathcal{S}^{n+1}(A)$ computable from index (i',m',j') to index $(\hat{i},\hat{m},\hat{j})$ observe that τ can be defined from L to G_φ, the former with index (i',m',j') and the latter with index $(\hat{i},\hat{m},\hat{j})$, and that relative to these indices τ is $\mathcal{S}^{n+1}(A)$ computable by applications of Corollary 4.8 as above. The identity isomorphism in question is a restriction of τ to $G_\varphi < L$ so is $\mathcal{S}^{n+1}(A)$ computable. The identity isomorphism from index $(\hat{i},\hat{m},\hat{j})$ to (i',m',j') is $\mathcal{S}^{n+2}(A)$ computable since if $G_\varphi = F/K$ for F f.g. free, the quotient map $F \to G_\varphi$, where G_φ has index (i',m',j'), is $\mathcal{S}^{n+2}(A)$ computable by Corollary 3.5. Then $x \in \hat{i}(G_\varphi)$ can be viewed as an element in F and so the identity is $\mathcal{S}^{n+2}(A)$ computable. This also shows that $G \to G_\varphi$, with respect to $(\hat{i},\hat{m},\hat{j})$, is an $\mathcal{S}^{n+2}(A)$ embedding since with respect to (i',m',j') it is an $\mathcal{S}^{n+1}(A)$ embedding which may be carried over to the new index by the $\mathcal{S}^{n+2}(A)$ computable identity isomorphism. □

Using the above we obtain a version of the Higman, Neumann, Neumann Theorem.

Theorem 5.5: Let G be an $\mathcal{S}^n(A)$ group for $n \geq 3$. Then there exists a group G' on three generators and a group G'' on two generators such that $G < G' < G''$ and:

1) G' is $\mathcal{S}^{n+1}(A)$ and the embedding $G \to G'$ is an $\mathcal{S}^{n+1}(A)$ embedding

ii) G' is $\mathcal{S}^{n+2}(A)$ standard and the embedding $G \to G'$ is an $\mathcal{S}^{n+2}(A)$ embedding with respect to any standard index of G'

iii) G'' is $\mathcal{S}^{n+3}(A)$ and the embedding $G \to G''$ is an $\mathcal{S}^{n+3}(A)$ embedding

iv) G'' is $\mathcal{S}^{n+3}(A)$ standard and the embedding $G \to G''$ is $\mathcal{S}^{n+3}(A)$ computable and an $\mathcal{S}^{n+4}(A)$ embedding with respect to any standard index of G''.

<u>Proof</u>: Statements i) and iii) are relativized versions of Theorem 3.2 and Corollary 3.2.1 of [4]. The relativized proofs hold. Statement ii) is a consequence of i), Corollary 3.5, and the observation that the identity isomorphism on a f.g., $\mathcal{S}^n(A)$ group is $\mathcal{S}^{n+1}(A)$ computable between the original index and the standard index of Corollary 3.5 and also from the standard index to the original index. Statement iv) follows immediately from ii) and Corollary 5.4 using the same construction as in the proof of Corollary 3.2.1 of [4] □

As before, the specialization of the above results to the case of $\mathcal{S}^3(A)$ groups for A $\mathcal{S}^n$ decidable, is of interest.

<u>Proposition 5.6</u> If G is $\mathcal{S}^3(A)$ for A $\mathcal{S}^n$ decidable, $n \geq 4$, and φ is an $\mathcal{S}^{n-1}(A)$ isomorphism in G, G_φ is $\mathcal{S}^n$. If G is f.g. and $\mathcal{S}^3(A)$ standard for A $\mathcal{S}^n$ decidable, $n \geq 5$, and φ is an $\mathcal{S}^{n-2}(A)$ isomorphism in G, G_φ is $\mathcal{S}^n$ standard.

<u>Proof</u>: Immediate from Theorem 5.3 and Corollary 5.4. □

<u>Proposition 5.7</u>. If G is $\mathcal{S}^3(A)$ for A $\mathcal{S}^n$ decidable then:

i) if $n \geq 4$, G can be embedded in an $\mathcal{S}^n$ group on three generators by an $\mathcal{S}^n$ embedding;

ii) if $n \geq 5$, G can be embedded in an $\mathcal{S}^n$ standard group on three generators by an $\mathcal{S}^n$ embedding;

iii) if $n \geq 6$, G can be embedded in an $\mathcal{S}^n$ group having two generators by an $\mathcal{S}^n$ embedding;

iv) if $n \geq 7$, G can be embedded in an $\mathcal{S}^n$ standard group having two generators by an $\mathcal{S}^n$ embedding.

<u>Proof</u>: Immediate from Theorem 5.5. □

In the remainder of this section we develop technical ideas for later application. Again, the casual reader may not wish to read furthe.. First we obtain an analog of Lemma 4.20 for the strong Britton extensions (see [4] Definition 3.3 and Lemma 3.1).

<u>Definition 5.8</u>: Let G be an $\mathcal{S}^n(A)$ group, φ an $\mathcal{S}^n(A)$ isomorphism in G with domain H, and $K < G$ an $\mathcal{S}^n(A)$ decidable subgroup of G for $n \geq 3$. We say K is <u>$\mathcal{S}^n(A)$ invariant under φ</u> if

(i) K is H, $\mathcal{S}^n(A)$ compatible

(ii) K is $\varphi(H)$, $\mathcal{S}^n(A)$ compatible

(iii) $\varphi(H \cap K) = \varphi(H) \cap K$.

Condition (iii) of the above definition says $h \in H \cap K$ iff $\varphi(h) \in K$. We can locate $\delta^{n+2}(A)$ decidable subgroups of G_φ (with normal form index) according to the following

<u>Lemma 5.9</u>. Let G be an $\delta^n(A)$ group for $n \geq 3$, φ an $\delta^n(A)$ isomorphism in G, $K<G$ an $\delta^n(A)$ decidable subgroup $\delta^n(A)$ invariant under φ. Define $H = \text{domain}\ \varphi$ and $\varphi' = \varphi\,|_{H\cap K}$. Then the embedding of the $\delta^{n+2}(A)$ group $K_{\varphi'}$ into the $\delta^{n+2}(A)$ group G_φ by $k \in K$, $k \mapsto k \in G$ and $t \mapsto t$ is an $\delta^{n+2}(A)$ embedding relative to the indices given by Theorem 5.3.

<u>Proof</u>. We proceed exactly as in the proof of Lemma 3.1 of [4]. Using the notation of that proof, we see $K_{\varphi'} < L'$ is an $\delta^{n+2}(A)$ decidable subgroup of the $\delta^{n+1}(A)$ group L' corresponding to some choice of $\delta^n(A)$ coset representative system, $L' < L$ is $\delta^{n+1}(A)$ decidable for an index on L given by a specific choice of coset representative system and by Corollary 4.9 the identity isomorphism on L is δ^{n+2} for any choices of $\delta^n(A)$ coset representative systems. Thus $K_{\varphi'} < L' < L \overset{id}{\cong} L > G_\varphi$ is an $\delta^{n+2}(A)$ embedding. The verification of the appropriate compatibility conditions for $L' < L$ is identical to that given in the cited proof noting that the decisions involved are all $\delta^n(A)$ decidable. □

We observe that when the conditions of Lemma 5.9 hold, $K_{\varphi'} = \{K,t\} < G_\varphi$ and, by Proposition 3.3 of [4], $\{K,t\} \cap G = K$ in G_φ.

Next, we consider a many-fold application of the strong Britton extension. In particular we allow countably many such extensions

corresponding to isomorphisms $\varphi_1, \varphi_2, \cdots$ in G. Observe that $G_{\varphi_1,\varphi_2\cdots}$ may be finitely or (countably) infinitely generated. Here each φ_i is to be an $\delta^n(A)$ isomorphism in G and not (more generally) in $G_{\varphi_1,\cdots,\varphi_{i-1}}$ so our result does not apply as it stands to the general case of "Britton towers".

<u>Lemma 5.10</u>. Let G be an $^n(A)$ group for $n \geq 3$ and $\varphi_1, \varphi_2, \cdots$ be an ordered sequence of $\delta^n(A)$ isomorphisms in G with domain $\varphi_k = H_k < G$. Define $G_\infty = G_{\varphi_1,\varphi_2\cdots} = \langle G, t_1, t_2 \cdots ; t_k h_k t_k^{-1} = \varphi_k(h_k)$ for all k, and $h_k \in H_k \rangle$. Then G_∞ is an $\delta^{n+2}(A)$ group and the embedding $G < G_\infty$ is an $\delta^{n+2}(A)$ embedding.

<u>Proof:</u> Our proof is similar to that of Lemma 3.2 of [4] but with two modifications (note that the functions $\sigma, \sigma_1, \sigma_2$ of [4] are our J, K, L).

Let $G = G_0$, $G_{\varphi_1,\cdots,\varphi_m} = G_m$. We first show that each G_m is an $\delta^{n+2}(A)$ group (clear for G_0). Let G' be a copy of G with $g' \in G'$, the copy of $g \in G$. Define $L_m = \left(G * \langle r_1, \cdots, r_m ; \rangle\right) *_\phi \left(G' * \langle s_1, \cdots, s_m ; \rangle\right)$ for $\phi : G * r_1 H_1 r_1^{-1} * \cdots * r_m H_m r_m^{-1} \to G' * s_1 \varphi_1(H_1)' s_1^{-1} * \cdots * s_m \varphi_m(H_m)' s_m^{-1}$ by $g \mapsto g'$ and $r_i h r_i^{-1} \mapsto s_i \varphi_i(h)' s_i^{-1}$ for all $h \in H_i$ and $i = 1, \cdots, m$. The factors of L_m are $\delta^n(A)$ groups by Corollary 4.7 and the domain and range of ϕ are $\delta^n(A)$ decidable by repeated application of the relativized version of Proposition 2.1 of [4]. It is clear that the individual homomorphisms $g \mapsto g'$ and $r_i h r_i^{-1} \mapsto s_i \varphi(h)' s_i^{-1}$ are $\delta^n(A)$ computable

(as homomorphisms from G or $r_i H_i r_i$ into $G' * \langle s_1, \cdots, s_m; \rangle$). Moreover since the normal form of $\Phi(w)$ for $w \in G * \langle r_1, \cdots r_n; \rangle$ is the same as that for w with only the symbols changed, it is clear that the recursion used to define Φ as an extension of these individual homomorphisms by Corollary 4.8 can be bounded so Φ is $\mathcal{E}^n(A)$ computable. Similarly Φ^{-1} is $\mathcal{E}^n(A)$ computable so L_m is an $\mathcal{E}^{n+1}(A)$ group by Theorem 4.6. The homomorphism $\tau: L_m \to L_m$ given by $g \mapsto g$, $g' \mapsto g'$, $r_i \mapsto s_i^{-1} r_i$ and $s_i \mapsto \{1\}$ is $\mathcal{E}^{n+2}(A)$ computable, fixing only $G_m < L_m$ by an argument entirely analagous to that used in showing τ $\mathcal{E}^{n+1}$ computable in the proof of Theorem 3.5 of [A]. Thus G_m is an $\mathcal{E}^{n+2}(A)$ group as an $\mathcal{E}^{n+2}(A)$ decidable subgroup of L_m. Let L_m (and hence G_m) have index (i_m, m_m, j_m).

Next we show that for $k<m$, the embedding $G_k < G_m$ is an $\mathcal{E}^{n+2}(A)$ embedding. This is immediate for $k=0$ and for $k>0$ it suffices to show $L_k < L_m$ is an $\mathcal{E}^{n+2}(A)$ embedding. Observe that $G * r_1 H_1 r_1^{-1} * \cdots * r_k H_k r_k^{-1} < G * r_1 H_1 r_1^{-1} * \cdots * r_m H_m r^{-1}$ and $G' * s_1 \varphi_1(H_1)' s_1^{-1} * \cdots * s_k \varphi_k(H_k)' s_k^{-1} < G' * s_1 \varphi_1(H_1)' s_1^{-1} * \cdots * s_m \varphi_m(H_m)' s_m^{-1}$ are $\mathcal{E}^3$ decidable and so the conditions of Lemma 4.20 are satisfied and $L_k < L_m$ is an $\mathcal{E}^{n+2}(A)$ embedding.

To form $i(G_\omega)$ let $n(g)$ = minimum m such that $g \in G_m$ and set $i(g) = J\big(n(g), i_{n(g)}(g)\big)$. Then

$$x \in i(G_\omega) \leftrightarrow L(x) \in i_{K(x)}\big(G_{K(x)}\big) \wedge \big(K(x) > 0 \to L(x) \notin i_{K(x)}\big(G_{K(x)-1}\big)\big)$$

so $i(G_\omega)$ is $\mathcal{E}^{n+2}(A)$ decidable. The inverse operation on the group G_ω is encoded by $j(x) = J\big(K(x), \bar{i}_{K(x)}\big(L(x)\big)\big)$ so j is $\mathcal{E}^{n+2}(A)$ computable. To define the encoded group multiplication, let $\kappa_{k,m}: i_k(G_k) \to i_m(G_m)$ be the $\mathcal{E}^{n+2}(A)$ embedding for $k<m$. Since $g \in G_k$ and $\tilde{g} \in G_m$ for $k<m$ implies $g\tilde{g} \in G_m$ and $g\tilde{g} \notin G_{m-1}$ we define

$$m(x,y) = \begin{cases} J\big(K(x), m_{K(x)}\big(L(x), L(y)\big)\big) & \text{if } K(x) = K(y), \\ J\big(K(x), m_{K(x)}\big(L(x)\, \kappa_{K(y), K(x)} L(y)\big)\big) & \text{if } K(x) > K(y), \\ J\big(K(y), m_{K(y)}\big(\kappa_{K(x), K(y)} L(x), L(y)\big)\big) & \text{if } K(x) < K(y), \end{cases}$$

so m is $\mathcal{E}^{n+2}(A)$ computable.

Clearly the embedding $G = G_0 < G_\omega$ is an $\mathcal{E}^{n+2}(A)$ embedding. □

In the above proof observe also that the embeddings $G_m < G_\omega$ are $\mathcal{E}^{n+2}(A)$ embeddings since the $G_k < G_m$ for $k<m$ are $\mathcal{E}^{n+2}(A)$ embeddings and for $x \in i_m(G_m)$ the computation of the minimal k such that $x \in i_m(G_k)$ can be performed by bounded minimalization and hence is $\mathcal{E}^{n+2}(A)$ computable.

We have the following extension to Lemma 5.9.

<u>Lemma 5.11</u>: Under the assumptions of Lemma 5.10, assume also that $K<G$ is an $\mathcal{E}^n(A)$ decidable subgroup which is $\mathcal{E}^n(A)$ invariant under the φ_k for all k. Then

i) $K_\omega = \langle K, t_1, \cdots; t_k h t_k^{-1} = \varphi_k(h)$ for all $k = 1, \cdots$ and all $h \in H_k \cap K\rangle = \{K, t_1, \cdots\} < G_\omega$ is an $\mathcal{E}^{n+2}(A)$ embedding

ii) $K_\omega \cap G = K$.

Proof: We use the notation in the proof of Lemma 5.10. It suffices to show the embedding $K_m = \langle K, t_1, \cdots, t_m; t_k h t_k^{-1} = \varphi_k(h)$ for $k = 1, \cdots, m$ and all $h \in H_k \cap K\rangle < G_m$ is an $\mathcal{E}^{n+2}(A)$ embedding. Define $L'_m = \left(K * \langle r_1, \cdots, r_m;\rangle\right) *_{\psi'} \left(K' * \langle s_1, \cdots, s_m;\rangle\right)$ for K' a copy of K, by $k \mapsto k'$ and

$$\psi' : K * r_1 H_1 \cap K r_1^{-1} * \cdots * r_m H_m \cap K r_m^{-1} \to K' * s_1 \varphi_1(H_1 \cap K)' s_1^{-1} * \cdots * s_m \varphi_m (H_m \cap K)' s_m^{-1}$$

by $k \mapsto k'$ and $r_i h r_i^{-1} \mapsto s_i \varphi_i(h)' s_i^{-1}$ for $k \in K$ and $h \in H_i \cap K$. Then as in the case of L_m, L'_m is an $\mathcal{E}^{n+1}(A)$ group. It suffices to show the embedding $L'_m < L_m$ is an $\mathcal{E}^{n+2}(A)$ embedding. This is immediate from Lemma 4.20 observing that

$$K * r_1 H_1 \cap K r_1^{-1} * \cdots * r_m H_m \cap K r_m^{-1} < G * r_1 H_1 r_1^{-1} * \cdots * r_m H_m r_m^{-1}$$

and

$$K' * s_1 \varphi_1 (H_1 \cap K)' s_1^{-1} * \cdots * s_m \varphi_m (H_m \cap K)' s_m^{-1} < G' * s_1 \varphi_1 (H_1)' s_1^{-1} * \cdots * s_m \varphi_m (H_m)' s_m^{-1}$$

are $\mathcal{E}^n(A)$ decidable and ψ' is the restriction of ψ. It is obvious that $K_m = \{K, t_1, \cdots, t_m\} < G_m$ and so $K_\omega = \{K, t_1, \cdots\} < G_\omega$ proving (i).

To prove (ii) it suffices to show $K_m \cap G = K$ for all m. By Lemma 4.20, $K_m \cap G < L'_m \cap G = L'_m \cap G \cap \left(G * \langle r_1, \cdots, r_m;\rangle\right) = \left(K * \langle r_1, \cdots, r_m;\rangle\right) \cap G = K$. Since $K_m > K$ and $G > K$, $K_m \cap G > K$ proving $K_m \cap G = K$ for all m. □

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